# Section: Eigenvalues and Eigenvectors

Applications (For next 8-9 lectures)

- Dynamical systems: predator/prey, RLC & SPRING/MASS/DAMPER
- Control systems
- Markov processes, population dynamics, markov chains and baseball statistics
- Opinion dynamics in social media

Topics:

- Linear dynamical systems (ALA 8.1)
- Determinants (ALA 1.9, abridged)
- Eigenvalues and eigenvectors (ALA 8.2)

Additional reading: ALA 5.1 and 5.2.

## Linear Dynamical Systems (ALA 8.1)

A dynamical system refers to the differential equations governing the change of a system over time. This system can be mechanical, electrical, fluid, biological, financial, or even social. In this next part of the class, we will build up techniques for solving differential equations describing linear dynamical systems. Two key mathematical tools used in extending our understanding of scalar dynamical systems, which you should have seen before in Math 1400/1410, to vector valued dynamical systems are eigenvalues and eigenvectors.

### Scalar Ordinary Differential Equations

Let's remind ourselves of the solution to first order scalar ordinary differential equations (ODEs), which take the form

$$\frac{du}{dt} = au, \qquad (S)$$

where $a \in \mathbb{R}$ is a known constant, and $u(t)$ is an unknown scalar function.

> **NOTATION**
>
> Note that you will sometimes see $\dot{u}$ instead of $\frac{du}{dt}$: the former is Newton's notation, and is commonly used when the argument of differentiation is time, whereas the latter is Leibniz's notation, and is commonly used to specify the argument of differentiation. Also note that equation (S) really means
>
> $$\frac{d}{dt} u(t) = a\,u(t),$$
>
> however, the argument $t$ of $u(t)$ is often omitted to make things less cumbersome to write.

The general solution to (S) is an exponential function

$$u(t) = ce^{at}, \qquad (\text{SOL})$$

where the constant $c \in \mathbb{R}$ is uniquely determined by the initial condition $u(t_0) = b$ (note we'll often take $t_0 = 0$ to keep things simple). Substituting $t = t_0$ into (SOL), we see that

$$u(t_0) = ce^{at_0} = b$$

so that $c = be^{-at_0}$, allowing us to conclude that

$$u(t) = be^{a(t-t_0)}$$

solves (S).

## Example: Half-Life of an Isotope

The radioactive decay of uranium-238 is governed by the differential equation

$$\frac{du}{dt} = -\gamma u.$$

Here $u(t)$ is the amount of U238 remaining at time $t$, and $\gamma > 0$ is the decay rate. The solution is

$$u(t) = c e^{-\gamma t},$$

where $c = u(0)$ is the initial amount of U238 at $t_0 = 0$. We see that the amount $u(t)$ is decaying to zero exponentially quickly with "rate" $\gamma$.

An isotope's half-life $t_*$ is how long it takes for the amount of a sample to decay to half its initial value, i.e., $u(t_*) = \frac{1}{2} u(0)$. To determine $t_*$, we solve

$$u(t_*) = u(0) e^{-\gamma t_*} = \frac{1}{2} u(0)$$

$$\iff e^{-\gamma t_*} = \frac{1}{2} \iff t_* = \frac{\log 2}{\gamma}.$$

Before proceeding to the general case, we make some simple but useful observations:

- The zero function $u(t) = 0 \ \forall t$ is a solution (sol) w/ $c = 0$. This is known as an equilibrium or fixed point solution.
- If $a > 0$, then solutions grow exponentially: this implies $u = 0$ is an unstable equilibrium, because any small nonzero initial condition $u(t_0) = \varepsilon$ will "blow up" far away from $u = 0$.
- If $a < 0$, the solutions decay exponentially: this implies $u = 0$ is a stable equilibrium (in fact globally asymptotically so), which means that $u(t) \longrightarrow 0$ as $t \to \infty$ for any initial condition $u(t_0)$.
- The borderline case is $a = 0$, in which case all solutions (sol) are constant, i.e., $u(t) = u(t_0)$ for all $t$. Such systems are called marginally stable (or just stable) because while they don't blow up on you, they also don't converge to $u = 0$.

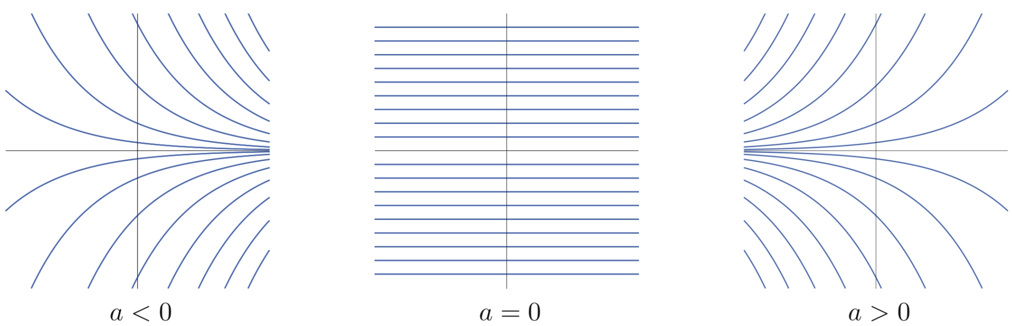


**Figure 8.1.** Solutions to $\dot{u} = a\,u$.

# First Order Dynamical Systems

We will concentrate most of our attention on homogeneous linear time-invariant first order dynamical systems. In this case, we have a vector valued solution

$$\underline{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$$

which parameterizes a curve in $\mathbb{R}^n$. This solution $\underline{u}(t)$ is assumed to obey a differential equation of the form:

$$\left.\begin{aligned} \frac{du_1}{dt} &= a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n \\ \frac{du_2}{dt} &= a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n \\ &\vdots \qquad\qquad\qquad \vdots \\ \frac{du_n}{dt} &= a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n \end{aligned}\right\}$$

or more compactly

$$\frac{d\underline{u}}{dt} = A\underline{u} \qquad \text{(LTI)}$$

for $A \in \mathbb{R}^{n\times n}$ a known constant matrix.

The question now becomes: what are the solutions to (LTI)? Well, let's take inspiration from the scalar solution (SOL) and investigate if and when

$$\underline{u}(t) = e^{\lambda t}\underline{v} \qquad \text{(GUESS)}$$

is a solution to (LTI). Here, $\lambda \in \mathbb{R}$ is a constant, as is $\underline{v} \in \mathbb{R}^n$. In other words, the components $u_i(t) = e^{\lambda t} v_i$ of (GUESS) are constant multiples of the same exponential function.

First, we compute the time derivative of (GUESS):

$$\frac{d}{dt}\underline{u}(t) = \frac{d}{dt}(e^{\lambda t}\underline{v}) = \lambda e^{\lambda t}\underline{v}.$$

Next, we compute the RHS of (LTI) with $\underline{u}(t)$ as in (GUESS):

$$A\underline{u} = A(e^{\lambda t}\underline{v}) = e^{\lambda t}(A\underline{v}) \qquad (e^{\lambda t} \text{ is a scalar}).$$

Therefore (GUESS) solves (LTI) if and only if $\lambda e^{\lambda t}\underline{v} = e^{\lambda t}A\underline{v} \iff \lambda\underline{v} = A\underline{v}$. This system of $n$ algebraic equations will be the topic of study for the next few lectures.

## Eigenvalues and Eigenvectors

Motivated by the discussion above, which we will return to, we define two fundamental elements of linear algebra: the eigenvalue and eigenvector.

For $A \in \mathbb{R}^{n\times n}$, a scalar $\lambda$ is called an eigenvalue of $A$ if there is a <u>nonzero</u> vector $\underline{v} \in \mathbb{R}^n$, called an eigenvector, such that

$$A\underline{v} = \lambda \underline{v}. \quad \text{(EIG)}$$

Geometrically, when $A$ acts on an eigenvector $\underline{v}$, it does not change its orientation: it only stretches it by the value specified by the eigenvalue $\lambda$.

The question then becomes how do we find eigenvalues and eigenvectors for a given matrix $A$? Now, if we knew $\lambda$, then (EIG) is a linear system in $\underline{v}$: indeed, we could solve the homogeneous linear system $(A-\lambda I)\underline{v} = \underline{0}$. We've already seen that the solution set is precisely the null space of $A-\lambda I$, i.e., $(A-\lambda I)\underline{v} = \underline{0}$ if and only if $\underline{v} \in \text{null}(A-\lambda I)$. We are interested in $\underline{v} \neq \underline{0}$, and we know that this can only occur if $A-\lambda I$ is singular! This discussion is summarized in the following theorem:

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Theorem: A scalar $\lambda$ is an eigenvalue of $A \in \mathbb{R}^{n\times n}$ if and only if the matrix $A-\lambda I$ is singular, i.e., $\text{rank}(A-\lambda I) < n$. The corresponding eigenvectors are the nonzero solutions to the eigenvalue equation $(A-\lambda I)\underline{v} = \underline{0}$

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This theorem gives us a plan of attack! First, find a scalar $\lambda$ such that $A-\lambda I$ is singular, and then use row reduction to solve $(A-\lambda I)\underline{v} = \underline{0}$. We know how to deal with the second step, but what about the first? For this, we will rely on the determinant.

## Determinants (ALA 1.9)

We assume that you have already seen determinants in Math 1410, and focus here on some key properties that will be needed for this section. Before proceeding, we pause to note that determinants have very deep meanings, especially in differential calculus, as they keep track of volumes as they are transformed via (linear or otherwise) functions. They are indeed very useful theoretical tools, but much like matrix inverses, are rarely computed by hand, except for $2\times 2$ cases.

Fact 1: The determinant of a matrix $A$, written $\det A$ or $|A|$, is only defined if $A$ is square.

Fact 2: The determinant of a $1\times 1$ matrix $A = [a]$ is $\det[a] = a$.
The determinant of a $2\times 2$ matrix is $\det\begin{bmatrix} a & b \\ c & d\end{bmatrix} = ad - bc$.

You may recognize this expression from our formula for the inverse of a 2x2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

In this case, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ exists if and only if $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad-bc \neq 0$.

This observation is true in general

<u>Fact 3</u>: $A^{-1}$ exists, i.e., $A$ is nonsingular, if and only if, $\det A = 0$.

A corollary of Fact 3, which we will use in our eigenvalue computations, is that $A$ is singular if and only if $\det A = 0$.

This is all we need for now to get started: in the online notes, we have a small aside / case study on computing determinants for large matrices using the QR factorization — this will not be tested, but highlights how useful the QR factorization of a matrix is!

<u>Back to Eigenvalues</u>

Using our corollary to Fact 3 above, we conclude that $\lambda$ is an eigenvalue of the matrix $A$ if and only if $\lambda$ is a solution to the characteristic equation

$$\det(A - \lambda I) = 0.$$

We now have all of the pieces we need to find eigenvalues and eigenvectors for 2x2 matrices (which is all we will ever ask you to compute by hand, unless there is special structure).

<u>Example</u>: Consider the 2x2 matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. We compute the determinant of $A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$ as $\det(A-\lambda I) = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8$.

Setting this to zero we see that $\det(A - \lambda I) = 0$ if and only if

$$\lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2) = 0,$$

i.e., if and only if $\lambda = 4$ or $\lambda = 2$. This means $A$ has two eigenvalues, which we denote $\lambda_1 = 4$ and $\lambda_2 = 2$. Next, for each eigenvalue, we solve $(A - \lambda_i I)\underline{v}_i = 0$.

$$\lambda_1: \ (A - 4I)\underline{v} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -v_1 + v_2 = 0 \\ v_1 - v_2 = 0 \end{matrix} \Rightarrow v_1 = v_2 = a \Rightarrow \underline{v} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector associated with $\lambda_1$ for any $a \neq 0$. We typically only distinguish linearly independent eigenvectors, so we would say that $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is <u>the</u> eigenvector associated with $\lambda_1 = 4$, although it is understood that any other $\tilde{\underline{v}}_1 = a\underline{v}_1$, for $a \neq 0$, is also a valid eigenvector.

$\lambda_2$: $(A-2I)\underline{v} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 + v_2 = 0 \Rightarrow v_1 = -v_2 = a \Rightarrow \underline{v} = a\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

We set $a=1$ to pick $\underline{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as the eigenvector associated with $\lambda_2 = 2$.

Therefore, the complete list of eigenvalue/vector pairs are

$$\lambda_1 = 4,\ \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda_2 = 2,\ \underline{v}_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We now introduce another determinant fact that will let us explore some other properties of eigenvalues without having to grind through pages of algebra:

__Fact 4__ If $U$ is a block upper triangular, i.e., if $U = \begin{bmatrix} U_{11} & U_{21} \\ 0 & U_{22} \end{bmatrix}$, for $U_{ij}$ of compatible dimension, then $\det U = \det U_{11} \cdot \det U_{22}$.

i.e., the determinant is given by the product of the determinants of its block diagonals.

__Example__: Consider the 3x3 matrix $A = \left[\begin{array}{c|cc} 2 & -1 & -1 \\ \hline 0 & 3 & 1 \\ 0 & 1 & 3 \end{array}\right]$. To compute its eigenvalues, we solve

$$\det(A-\lambda I) = \det\left[\begin{array}{c|cc} 2-\lambda & -1 & -1 \\ \hline 0 & 3-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{array}\right] \overset{\text{Fact 4!}}{=} \det[2-\lambda]\det\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$
$$= (2-\lambda)(\lambda-4)(\lambda-2) = 0$$

Most 3x3 matrices have three distinct eigenvalues, but this one only has two: $\lambda_1 = 2$, which is a double eigenvalue, as it is a double root of (*), along with a simple eigenvalue $\lambda_2 = 4$. The eigenvector equation for $\lambda_1 = 2$ is

$$(A-2I)\underline{v} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_2 + v_3 = 0 \text{ and } v_1 \text{ free}$$

i.e., $\underline{v} = \begin{bmatrix} a \\ b \\ -b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

This solution depends on two free variables, $a$ and $b$, and so any nonzero linear combination of the "basis eigenvectors" $\underline{v}_1 = (1,0,0)$ and $\hat{\underline{v}}_1 = (0,1,-2)$ is a valid eigenvector.

On the other hand, the eigenvector equation for the simple eigenvalue $\lambda_2 = 4$ is

$$(A-4I)\underline{v} = \begin{bmatrix} -2 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} 2v_1 + v_2 + v_3 = 0 \\ v_2 = v_3 \end{array} \quad \begin{array}{l} 2v_1 + 2v_2 = 0 \\ \Rightarrow v_1 = -v_2 = -v_3 \end{array}$$

So the general solution is $\underline{v} = a\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, and we designate $\underline{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ as the eigenvector associated with $\lambda_2 = 4$. In summary, the eigenvalues and "basis" eigenvectors for this matrix are:

$$\lambda_1 = 2, \qquad \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\underline{v}}_1 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

(EIGENBASIS)

$$\lambda_2 = 4, \qquad \underline{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

In general, given a real eigenvalue $\lambda$, the corresponding eigenspace $V_\lambda \subset \mathbb{R}^n$ is the subspace spanned by all its eigenvectors, or equivalently, the eigenspace is the null space

$$V_\lambda = \text{null}(A - \lambda I).$$

Thus, $\lambda \in \mathbb{R}$ is an eigenvalue if and only if $V_\lambda \neq \{0\}$, in which case any nonzero element of $V_\lambda$ is an eigenvector. Typically, we describe eigenspaces in terms of their basis elements, as we did in (EIGENBASIS).

This description gives us a very important connection between zero eigenvalues and the invertibility of a matrix $A$:

**Theorem:** $\lambda = 0$ is an eigenvalue if and only if $\text{null}(A) \neq \{0\}$, i.e., if and only if $A$ is singular.